Newton's Divided Difference Interpolation
&
Hermite Interpolation

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Overview

An approximating polynomial for a given function is discussed, called **Newton's divided differences interpolation polynomial.**

The coefficients of the polynomial are calculated using **divided differences.**

We discuss Newton's forward and backward divided differences.

We next discuss Hermite interpolation which helps us in finding an "approximate value of the given function" at a **special point**, from the available information of f and its derivative, at the special point.

Suppose that $P_n(x)$ is the nth Lagrange polynomial that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$.

Although this polynomial is unique, alternate algebraic representations are useful in certain situations.

The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ are used to express $P_n(x)$ in the form

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

for appropriate constants $a_0, a_1, \ldots, a_n$.

To determine the first of these constants, a_0, note that if $P_n(x)$ is written in the form of the above equation, then evaluating $P_n(x)$ at x_0 leaves only the constant term a_0. That is,

$$a_0 = P_n(x_0) = f(x_0).$$

Similarly, when $P(x)$ is evaluated at x_1, the only nonzero terms in the evaluation of $P_n(x)$ are the constant and linear terms,

$$f(x_0) + a_1(x_1 - x_0) = P_n(x_1) = f(x_1)$$

so

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Divided Difference Notation

The divided-difference notation, is introduced, which is similar to Aitken's Δ^2 notation.

The **zeroth divided difference** of the function f with respect to x_i, denoted by $f[a_i]$, is simply the value of f at x_i,

$$f[x_i] = f(x_i).$$

The remaining divided differences are defined inductively.

The **first divided difference** of f with respect to x_i and x_{i+1} is denoted by $f[a_i, x_{i+1}]$ and is defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}.$$

The **second divided difference**, $f[x_i, x_{i+1}, x_{i+2}]$, is defined as

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}.$$

The process ends with single nth **divided difference**,

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, x_2, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}.$$

Hence $P_n(x)$ cane be rewritten as

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \dots, x_k](x - x_0) \cdots (x - x_{k-1}).$$

The value of $f[x_0, x_1, \dots, x_k]$ is independent of the order of the numbers $x_0, x_1, \dots, x_k$.

Divided Difference Table

x	$f(x)$	First divided differences	Second divided differences
x_0	$f[x_0]$		
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$	
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$	
x_2	$f[x_2]$		

Suppose f is continuously differentiable on $[x_0, x_1]$. By the mean value theorem, there exists $\xi \in [x_0, x_1]$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1].$$

The following theorem generalizes this result.

Theorem

Suppose that $f \in C^n[a, b]$ and $x_0, x_1, \ldots x_n$ are distinct numbers in $[a, b]$. Then a number ξ (generally unknow) exists in (a, b) with

$$f[x_0, x_1, \ldots x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

Nodes with equal spacing :

Let $h = x_{i+1} - x_i$ for each $i = 0, 1, \ldots, n-1$. Hence

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\ h^k\ f[x_0, x_1, \ldots, x_k].$$

Newton Forward Difference

We use the forward difference notation Δ introduced in Aitken's Δ^2 method.

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i), \qquad \text{for } i \geq 0.$$

Higher powers are defined recursively by

$$\Delta^k f(x_i) = \Delta(\Delta^{k-1} f(x_i)) \quad \text{for } i \geq 0.$$

With this notation,

$$\begin{aligned} f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1}{h}\Delta f(x_0) \\ f[x_0, x_1, x_2] &= \frac{1}{2h}\left[\frac{\Delta f[x_1] - \Delta f[x_0]}{h}\right] = \frac{1}{2h^2}\Delta^2 f(x_0) \end{aligned}$$

and, in general

$$f[x_0, x_1, \ldots, x_k] = \frac{1}{k!h^k}\Delta^k f(x_0).$$

Newton Backward Difference

Backward difference operator is defined by

$$\nabla f(x_i) = f(x_i) - f(x_{i-1}), \qquad \text{for } i \geq 1.$$

Higher powers are defined recursively by

$$\nabla^k f(x_i) = \nabla(\nabla^{k-1} f(x_i)) \quad \text{for } i \geq 2.$$

With this notation,

$$\begin{aligned} f[x_n, x_{n-1}] &= \frac{1}{h}\nabla f(x_n) \\ f[x_n, x_{n-1}, x_{n-2}] &= \frac{1}{2h^2}\nabla^2 f(x_n) \end{aligned}$$

and, in general

$$f[x_n, x_{n-1}, \ldots, x_{n-k}] = \frac{1}{k!h^k}\nabla^2 f(x_n).$$

Hermite Interpolation

Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials.

Suppose that we are given $n+1$ distinct numbers $x_0, x_1, \ldots, x_n$ in $[a, b]$ and nonnegative integers $m_0, m_1, \ldots, m_n$, and

$$m = \max\{m_0, m_1, \ldots, m_n\}.$$

Note that the (unknown) function f is m_i-times differentiable at x_i.

The osculating polynomial approximating a function $f \in C^m[a, b]$ at x_i, for each $i = 0, 1, \ldots, n$, is the polynomial of least degree with the property that it agrees with the function f and all its derivatives of order less than or equal to m_i at x_i.

That is, the osculating polynomial $P(x)$ approximating a function $f \in C^m[a, b]$ satisfies the following :

For each $i = 0, 1, 2, \ldots, n$

1. $P(x_i) = f(x_i)$
2. $P^k(x_i) = f^k(x_i)$, for all $1 \leq k \leq m_i$.

$P(x)$ is **the unique polynomial of least degree with the above properties.**

Special Cases :

1. When $n = 0$, the osculating polynomial P approximating f is the m_0th Taylor polynomial for f at x_0.
2. When $m_i = 0$ for each i, the osculating polynomial P approximating f is the nth Lagrange interpolating polynomial for f at $x_0, x_1, \ldots, x_n$.

The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the **Hermite polynomials**.

For a given function f, these polynomials agree with f at $x_0, x_1, \ldots, x_n$.

In addition, since their first derivatives agree with those of f, they have the same "shape" as the function at $(x_i, f(x_i))$ in the sense that the **tangent lines** to the polynomial and to the function agree.

We restrict our attention to Hermite polynomials.

Theorem

If $f \in C^1[a,b]$ and $x_0, x_1, \ldots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, x_1, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} H_{n,j}(x)\ f(x_j) + \sum_{j=0}^{n} \tilde{H}_{n,j}(x)\ f'(x_j)$$

where

$$H_{n,j}(x) = \Big[1 - 2(x - x_j)L'_{n,j}(x_j)\Big] L^2_{n,j}(x) \text{ and } \tilde{H}_{n,j}(x) = \Big[(x - x_j)\Big] L^2_{n,j}(x).$$

Here $L_{n,j}(x)$ denotes the jth Lagrange polynomial of degree n,

$$L_{n,j}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}.$$

Error Term

$H_{2n+1}(x)$ is the Hermite polynomial of degree at most $2n+1$

1. agreeing with f at $x_0, x_1, \ldots, x_n$, and
2. their first derivatives (of $H_{2n+1}(x)$) agreeing with those of f.

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \cdots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi)$$

for some (generally unknow) ξ in the interval (a, b).

How to find Hermite polynomial?

For large value of n, the Hermite interpolation method is tedious to apply. An explanation is given for three nodes.

Suppose we are given a table containing values of the triplets

$$\left[x_k, f(x_k), f'(x_k)\right], \text{ for } k = 0, 1, 2.$$

Calculate the three Lagrange polynomials (each of degree 2) about

$$\{x_1, x_2\}, \ \{x_2, x_0\} \ \text{ and } \ \{x_0, x_1\},$$

denoted the polynomials by $L_{2,0}(x), \ L_{2,1}(x), \ L_{2,2}(x)$.

Calculate their derivates $L'_{2,0}(x), \ L'_{2,1}(x), \ L'_{2,2}(x)$.

How to find Hermite polynomial?

The polynomials

$$H_{2,0}(x),\ H_{2,1}(x),\ H_{2,2}(x)$$

and

$$\tilde{H}_{2,0}(x),\ \tilde{H}_{2,1}(x),\ \tilde{H}_{2,2}(x).$$

are calulated.

Hence the Hermite polynomial of degree 5

$$H_5(x) = H_{2,0}(x)\ f(x_0) + H_{2,1}(x)\ f(x_1) + H_{2,2}(x)\ f(x_2) + \tilde{H}_{2,0}(x)\ f'(x_0) + \tilde{H}_{2,1}(x)\ f'(x_1) + \tilde{H}_{2,2}(x)\ f'(x_2).$$

Finally, we can evaluate an "approximate value of f" at the specified point. Note that the Hermite polynomial H_5 agrees with f and its derivative, at the given nodes x_0, x_1, x_2.

References

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